

# Geometric Measure Theory and its Applications

4/23/2007

First, an addendum to last week's lecture: any two orthogonal, unit length vectors in some fixed plane give the same element of the space of 2-vectors when wedged together provided the ordered pair can be obtained from some fixed ordered pair by a rotation in that plane.

Example

$\vec{e}_1$  and  $\vec{e}_2$  orthonormal vectors

$\Rightarrow \{ \alpha \vec{e}_1 + \beta \vec{e}_2, -\beta \vec{e}_1 + \alpha \vec{e}_2 \}$  are also orthonormal if  $\alpha^2 + \beta^2 = 1$ .

$$\begin{aligned} (\alpha \vec{e}_1 + \beta \vec{e}_2) \wedge (-\beta \vec{e}_1 + \alpha \vec{e}_2) &= \alpha^2 (\vec{e}_1 \wedge \vec{e}_2) - \beta^2 (\vec{e}_2 \wedge \vec{e}_1) \\ &= (\alpha^2 + \beta^2) (\vec{e}_1 \wedge \vec{e}_2) \end{aligned}$$

This example adds the right correction or adjustment to the intuition that  $\vec{v}_1 \wedge \vec{v}_2$  "is" a little parallelogram with sides  $\vec{v}_1$  and  $\vec{v}_2$ . As  $\vec{v}_1 \wedge \vec{v}_2$  is really an oriented plane with mass or weight.

Of course, this comment generalizes to  $n$ -vectors and  $n$ -planes.

Mass, Compactness and representation by integration

- $|w|$  = euclidean norm of an  $n$ -covector

example 
$$\begin{aligned} &| \alpha e_1^* \wedge e_2^* + \beta e_3^* \wedge e_1^* + \gamma e_2^* \wedge e_3^* | \\ &= \sqrt{\alpha^2 + \beta^2 + \gamma^2} \end{aligned}$$

- $\sup_{x \in U} |w(x)|$  (which we call the mass norm on forms) is a sup norm on  $\mathcal{D}^n(U)$  we denote it by  $M(w)$  or  $\|w\|$ .  $\mathcal{D}^n(U)$  is separable with this norm.

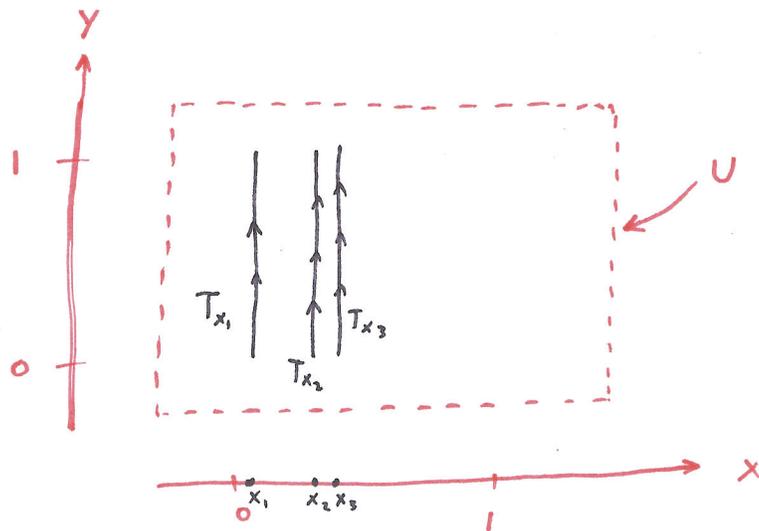
- Mass of  $T \in \mathcal{D}_n(U)$

$$M(T) \equiv \sup_{\omega \in \mathcal{D}^n(U)} T(\omega) \mid M(\omega) \leq 1$$

= operator norm on  $\mathcal{D}_n^*(U)$  using the mass norm on  $\mathcal{D}^n(U)$

$\mathcal{D}_n(U)$  is not separable under the mass norm.

Example:



For  $x \in [0, 1]$   $T_{\bar{x}}$  is the 1-current given by  $\text{supp}(T_{\bar{x}}) = \{(x, y) \mid x = \bar{x}, y \in [0, 1]\}$ , the orienting 1-vec  $\eta_{T_{\bar{x}}} = \bar{e}_2$ , the orthonormal vector parallel to the y axis, and  $\|T_{\bar{x}}\|$  is 1-dimensional Hausdorff measure restricted to the support of  $T_{\bar{x}}$  ( $= \text{supp}(T_{\bar{x}})$ ). For an arbitrary

$$\text{1-form } \omega = f_1 dx + f_2 dy$$

$$T_{\bar{x}}(\omega) = \int_0^1 f_2(\bar{x}, y) dy$$

$M(T_{x_1} - T_{x_2})$  (for  $x_1 \neq x_2$ ) = 2 (since one can easily construct a smooth 1-form which is 1 on  $T_{x_1}$  and -1 on  $T_{x_2}$ ).

We therefore have an uncountably infinite set of currents each of which lies in an open ball of radius 1 (using mass norm) which does not intersect any other unit balls around the other currents.

- currents with finite mass: If  $M(T) < \infty$  then a Riesz representation theorem tells us that for this  $T \in \mathcal{D}'_n(U)$  there is

$\eta_T$  :  $n$ -vec field

$\|T\|$  : Radon measure

This is "representation by integration" ~~of~~

← currents with finite mass are representable by integration

such that

$$T(w) = \int \langle \eta_T, w \rangle d\|T\|$$

Digression: what is a Radon measure?

$\mu$  (a measure) is

Regular: if  $\forall A \subset X \exists$  measurable  $B \ni A \subset B$  and  $\mu(A) = \mu(B)$

Borel Regular: if the "B" in "Regular" can be taken to be a Borel set.

Radon: if it is Borel Regular and  $\mu(K) < \infty$   $\forall$  compact sets in  $\mathbb{R}^n$ .

Theorem: if  $\mu$  is Radon on  $\mathbb{R}^n$

1)  $\forall A \subset \mathbb{R}^n, \mu(A) = \inf \{ \mu(U) \mid A \subset U, U \text{ open} \}$

2)  $\forall \mu$ -measurable  $A \subset \mathbb{R}^n, \mu(A) = \sup \{ \mu(K) \mid K \subset A, K \text{ compact} \}$

• Two topologies:

- we now have two topologies on  $\mathcal{D}^n(U)$ . The first is the topology generated by the seminorms  $p_i$ .

$$p_i(w) \equiv \sum_{k=0}^i \|D^k w\| \quad (\text{TOP 1})$$

If a linear functional on  $\mathcal{D}^n(U)$  is continuous w.r.t. this seminorm topology (TOP 1) then it is a member of  $\hat{\mathcal{D}}_n(U)$  (This is a definition of  $\hat{\mathcal{D}}(U)$ ).

The second topology on  $\mathcal{D}^n(U)$  is the supremum topology

$$\|w\| = \sup_{x \in U} |w(x)| \quad (\text{TOP 2})$$

A linear functional on  $\mathcal{D}^n(U)$  is continuous w.r.t. this topology exactly when  $T(w) \leq C \|w\|$  for some constant  $C$ . Clearly  $C$  is the mass of  $T$ .

Notice that  $T(w) \leq M(T) \cdot p_i(w)$ .

It is a fact that  $T \in \mathcal{D}_n(U) \Leftrightarrow T(w) < C p_i(w)$  for some  $i$  and all  $w \in \mathcal{D}^n(U)$ . ~~Therefore~~ therefore, we get that (abbreviated continuous linear functional to clf)

$$\text{clf}_{\text{Mass}}(\mathcal{D}^n) \subset \text{clf}_{\text{seminorm}}(\mathcal{D}^n) = \hat{\mathcal{D}}_n$$

- $\hat{\mathcal{D}}^n(U)$  is <sup>Mass</sup>dense in  $C_c(U)$ , the space of continuous functions with compact support.  $C_c(U)$  is separable as is  $\hat{\mathcal{D}}^n(U)$  with the mass norm.

- We have already noted that  $\mathcal{D}_n(U)$  is not separable w.r.t. the mass norm. What about its dual space

$(\mathcal{D}_n(U))^{\text{* mass}}$ ? (... mass indicating the mass norm of course)

Fact:  $X^*$  separable  $\Rightarrow X$  separable

Quick proof: choose dense  $\{f_i\}_{i=1}^\infty \subset X^*$ . choose  $\{x_i\}_{i=1}^\infty \subset X$   
 $\exists \|x_i\|=1$  and  $f_i(x_i) \geq \frac{1}{2} \forall i$ . If  $\{x_i\}_{i=1}^\infty$  is not dense in  $X$ , then define  $L \equiv$  closed linear span of  $\{x_i\}_{i=1}^\infty$ . Hahn-Banach says there is a continuous linear functional (i.e. element of  $X^*$ )  $\hat{f}$   
 $\exists \hat{f} = 0$  on  $L$  but  $\hat{f} \neq 0$ . Note that we have  $\sup_{\|x\|=1} |\hat{f} - f_i| \geq \frac{1}{2} \Rightarrow$   
 $\|\hat{f} - f_i\| \geq \frac{1}{2} \Rightarrow \{f_i\}_{i=1}^\infty$  is not dense in  $X^*$   
 $\Rightarrow \Leftarrow$ .

$\Rightarrow$  Since  $\mathcal{D}_n(U)$  is not separable w.r.t. mass norm, neither is  $(\mathcal{D}_n(U))^{\text{* mass}}$ .

- previous mistake: in lecture I said something about a family of  $\omega$ 's  $\subset (\mathcal{D}_n(U))^{\text{* mass}}$  that were not countable and ~~were~~ provided an example of an uncountable well separated (i.e. in a bunch of  $\epsilon$ -balls) that is nonsense, disjoint

these  $\omega$ 's live in  $\mathcal{D}^n(U)$  which is separable.  $(\mathcal{D}_n(U))^{\text{* mass}}$  is not separable because it is much bigger than  $\mathcal{D}^n(U)$  which it contains.  $\mathcal{D}^n(U) \subsetneq (\mathcal{D}_n(U))^{\text{* mass}}$

**Note:** when writing expressions like  $(\mathcal{D}_n(U))^{\text{* mass}}$  I am of course referring to the currents with bounded mass at the dual space of these etc.

silly recall diagram

iiii

- weak compactness

Alaoglu's theorem tells us that a mass bounded sequence of currents  $\{T_i\}_{i=1}^{\infty}$  is weak\* (pronounced "weak-star") convergent to some current  $T$ , after perhaps going to a subsequence.

I.e.

$$T_{i_k}(w) \rightarrow T(w) \quad \text{for some } T, \text{ for all } w \in \mathcal{D}^n(U) \\ \text{for some subsequence } i_k \text{ of } \mathbb{N}.$$

we denote this by  $T_{i_k} \xrightarrow{*} T$  or even  $T_{i_k} \rightarrow T$  (though this is rather careless)

Reminder: let  $X$  be a normed linear space with norm  $\|\cdot\|$ . Denote the dual space of cont. lin. functionals on  $X$  by  $X^*$ .

- $X_i \xrightarrow{i \rightarrow \infty} X$  ( $X_i$  converges strongly to  $X$ ) if  $\|X_i - X\| \rightarrow 0$
- $X_i \xrightarrow{i \rightarrow \infty} X$  ( $X_i$  converges weakly to  $X$ ) if  $f(X_i) \rightarrow f(X) \quad \forall f \in X^*$
- $X_i \xrightarrow{i \rightarrow \infty} X$  ( $X_i$  converges weak\* to  $X$ ) if  $g(X_i) \rightarrow g(X) \quad \forall g \in Y$   
where  $X = Y^*$   
(and so  $X^* = Y^{**} \supset Y$ )

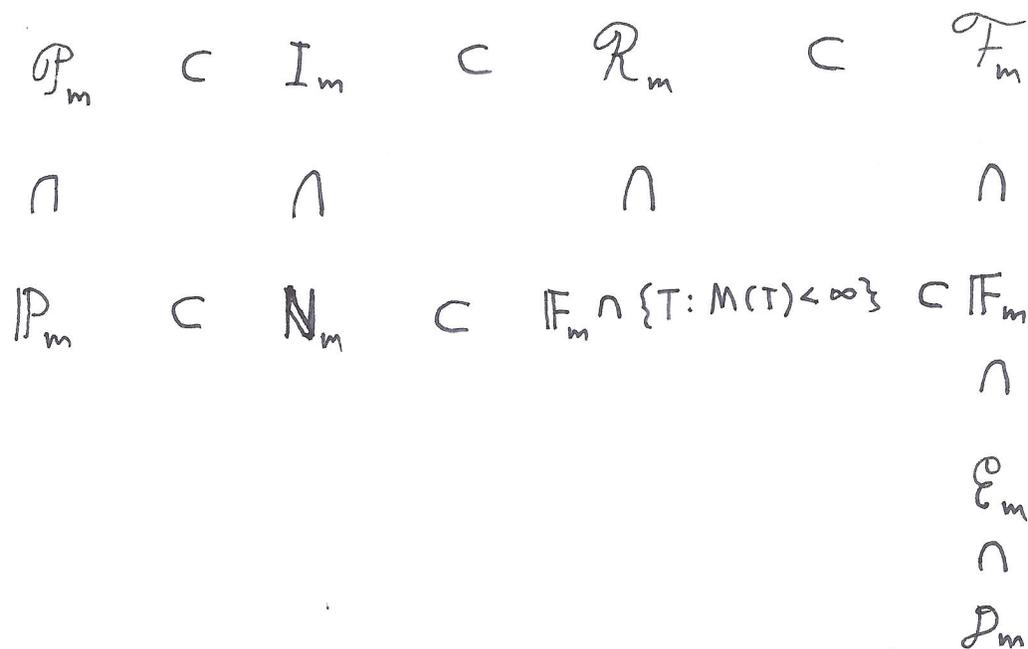
- Recall that  $\mathcal{D}^n(U)$  is mass separable while

$(\mathcal{D}^n(U))^{** \text{ mass}} = (\mathcal{D}_n(U))^{* \text{ mass}}$  is not  $\Rightarrow \mathcal{D}^n(U)$  is not reflexive, I.e.  $\mathcal{D}^n(U) \neq (\mathcal{D}^n(U))^{** \text{ mass}}$ . Therefore weak and weak\* convergence are not the same. Therefore my comments above about care (and the lack thereof!)

Now Back to Pictures and geometric things.

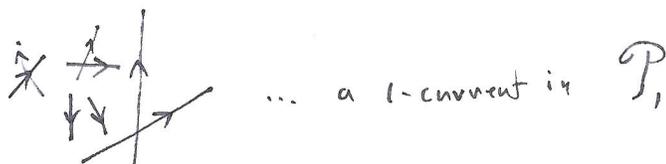
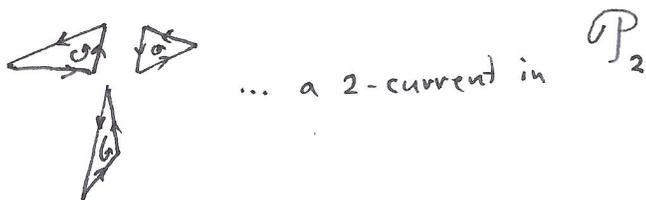
A Zoo of Currents: We now define 9 families or subclasses of currents. Examples of each are given. The relations between the families ~~are~~ are given.

First, a diagram (appears in both Morgan and Federer) of currents in  $\mathbb{R}^n$



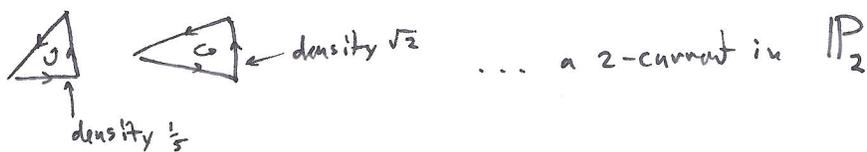
Definitions and examples:

$\mathcal{P}_m$ : Integral Polyhedral Chains = sums of oriented  $m$ -simplices in  $\mathbb{R}^n$  with integer densities



$\mathcal{P}_m$  are Abelian groups

$\mathbb{P}_m$  : real polyhedral chains = sums of oriented  $m$ -simplices in  $\mathbb{R}^n$  with real densities.



$\mathbb{P}_m$  are vector fields over the real numbers

Next we jump to  $\mathcal{R}_m$

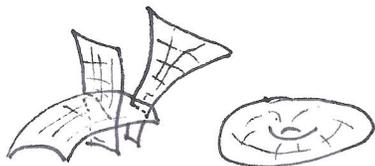
$\mathcal{R}_m$  : rectifiable currents = currents with integer densities whose support's are rectifiable sets. I.e.  $T \in \mathcal{R}_m$  iff

$$T(\omega) = \int_{\text{supp}(T)} \langle \omega, \eta \rangle d\mathcal{H}^m \llcorner (\text{supp}(T))$$

and  $\text{supp}(T)$  is an  $m$ -countably rectifiable set with compact support and mass:  $M(T) < \infty$ ,  $\text{supp}(T)$  compact



... a "nice" 1-current in  $\mathcal{R}_1$



... a 2-current in  $\mathcal{R}_2$

$\mathcal{R}_m$  is a group (more precisely,  $\mathcal{R}_{m,K}(U)$ , the rectifiable  $m$ -currents in  $U$  with support in  $K$ .  $\overset{C}{\text{compact}} U$  form an Abelian group)

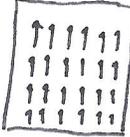
# Back to $I_m$

$I_m$ : Integral currents =  $T \in \mathcal{R}_m \ni \partial T \in \mathcal{R}_{m-1}$

The way I drew the currents illustrating above, they were clearly also in  $I_m$ : they have rectifiable boundaries.  $\mathcal{R}_m$



$N_m$ : Normal currents =  $T \ni M(T) < \infty$  and  $M(\partial T) < \infty$

$T =$ 

 $\left. \begin{array}{l} \|T\| = \mathcal{H}^2[0,1]^2 \\ \eta = \bar{e}_2 \\ \text{if } \omega = f dx + g dy \\ T(\omega) = \int_{[0,1]^2} g d\mathcal{H}^2 \end{array} \right\} \begin{array}{l} \in N_2 \\ \notin I_1 \end{array}$

Top dimensional currents

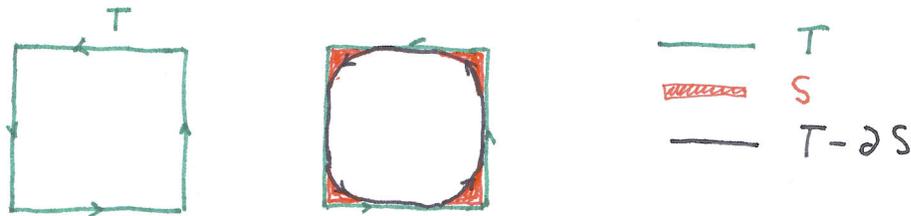
$T = f dx \wedge dy, f = \text{some smooth function from } \mathbb{R}^2 \rightarrow \mathbb{R} \text{ with compact support} \left. \right\} \in N_2 \text{ in } \mathbb{R}^2$   
 $T = f dx \wedge dy, f = \text{a BV function on } \mathbb{R}^2 \left. \right\} \in N_2 \text{ in } \mathbb{R}^2$

Normal currents often arise as **smoothings** of Integral currents.  
 (Imagine convolving some bump function with an integral current.)

To continue, we must introduce the flat norm  $\mathbb{F}$  for  $T \in \mathcal{D}_m$

$$\min_{S \in \mathcal{D}_{m+1}} \{M(T - \partial S) + M(S)\} = \sup_{\substack{\omega \in \mathcal{D}^m, \|\omega\| \leq 1, \|\partial \omega\| \leq 1 \\ \text{a theorem}^*}} T(\omega) \equiv \mathbb{F}(T)$$

Example



The above example shows the minimality decomposition of  $T$  into  $R = T - \partial S$  and  $\partial S$ .

Note on theorem\*

(whoops!  $\phi \equiv \omega$  throughout)  
(if that matters to you!)

write  $T$  as  $R + \partial S$ . then  $\sup_{\|\phi\| \leq 1, \|\partial \phi\| \leq 1} T(\phi)$

$$\text{is } \sup_{\|\phi\| \leq 1, \|\partial \phi\| \leq 1} \{R(\phi) + \partial S(\phi)\} = \sup_{\substack{\|\phi\| \leq 1 \\ \|\partial \phi\| \leq 1}} \{R(\phi) + S(\phi)\} \leq M(R) + M(S)$$

$$\Rightarrow \sup_{\substack{\omega \in \mathcal{D}^n, \|\omega\| \leq 1, \|\partial \omega\| \leq 1 \\ \text{take your pick } \left\{ \begin{array}{l} \omega \in \mathcal{D}^n, \|\omega\| \leq 1, \|\partial \omega\| \leq 1 \\ \phi \in \mathcal{D}^n, \|\phi\| \leq 1, \|\partial \phi\| \leq 1 \end{array} \right\}}} T(\phi) \text{ (or } T(\omega)) \leq \mathbb{F}(T)$$

To see the other direction ( $\geq$ ) see Federer 4.1.12

Now we are ready to define  $\mathbb{F}_m$

$\mathbb{F}_m$ : real flat chains or simply flat chains = The  $\mathbb{F}$  closure of  $\mathcal{N}_m$ .

Example

$$T = \text{circle} = \partial \left( \text{shaded disk} \right)$$

↑ a disk shaped region of  $\mathbb{R}^2$  with infinite length boundary and density  $\alpha \notin \mathbb{Z}$

Note:

$$T = \text{circle} = \text{circle} + \partial \left( \text{red shaded disk} \right)$$

$R + \partial S$  ( $\leftarrow$  both  $R, S$  have density  $\alpha \notin \mathbb{Z}$ )

So  $F(T - R) \leq M(S) = \text{red area}$  and we conclude that  $T$  is a finite flat distance from  $R$ , a normal current.

$\mathcal{F}_m$ : integral flat chains = m-currents  $T \ni T = R + \partial S$   
 $R \in \mathcal{R}_m, S \in \mathcal{R}_{m+1}$ .

Integral flat norm for  $T \in \mathcal{F}_m$

$$F(T) = \inf \{ M(R) + M(S) \mid T = R + \partial S, R \in \mathcal{R}_m, S \in \mathcal{R}_{m+1} \}$$

$$* = \inf \{ M(T - \partial S) + M(S) \mid S \in \mathcal{R}_{m+1} \}$$

... inf's can be replaced by min's since the inf are obtained.

To show \* you need to use the fact that  $S$  rect. and  $M(\partial S)$  finite  $\Rightarrow \partial S$  rect.

Example: use same example as above on this page but with  $\alpha \in \mathbb{Z}$ .

$\mathcal{E}_m$ : Currents with compact support

### Example

$T \in \mathcal{E}_1(\mathbb{R}^2)$  defined by  $T(fdx + gdy) = f''(0)$   
has compact support  $\{0\}$  but is not flat.

Proof

$$\begin{aligned} F(T) &= \sup \{ T(fdx + gdy) \mid \sqrt{f^2 + g^2} \leq 1, |g_x - f_y| \leq 1 \} \\ &= \sup \{ f''(0) \mid \sqrt{f^2 + g^2} \leq 1, |g_x - f_y| \leq 1 \} \\ &= \infty \end{aligned}$$

But if  $T \in \mathcal{F}_m$ ,  $F(T)$  is finite. (Recall the observation that if  $T = R + \partial S$   $F(T - R) \in M(S) < \infty$ )

$\mathcal{D}_m$ : Currents, the space dual to  $\mathcal{D}_m$ , ~~the~~ compactly supported  $m$ -forms.

### Example

$T \in \mathcal{D}_1(\mathbb{R}^2)$  defined by  $T(fdx + gdy) = f''(0,0) + f''(1,1) + f''(2,2) + f''(3,3) \dots$

That this current is not flat follows from the previous example. that the support is not compact is obvious.

**Full statement of earlier fact:** I said in these notes that  $T \in \mathcal{D}_m$  iff  $\exists i \ni T(\omega) < C P_i(\omega)$ ,  $P_i(\omega) \equiv \sum_{k=0}^i \|\partial^k \omega\|$ . Actually  $T \in \mathcal{D}_m^*$  iff for each compact set  $K \exists i_K \ni T(\omega) < C(K) P_{i_K}(\omega) \forall \omega$  supported in  $K$ . Since everything below  $\mathcal{D}_m$  in the lattice of currents has compact support this is a ~~minor~~ minor point.